Online Appendix for Reputation for Privacy

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Proof of Proposition 7. We assume without loss of generality that firm 1 is a rational firm. Denote the regulator's signal about whether firm i sold the data by $s_i^r \in \{+,-\}$, where $s_i^r = +$ means the regulator detects firm i's data sales (the regulator will charge the firm a constant fine f) and $s_i^r = -$ means the regulator does not detect data sales by firm i. The consumer can perfectly infer the regulator's signal according to whether the regulator fines a firm. So, we assume without loss of generality that the consumer also observes s_i^r to avoid introducing another notation about whether the regulator fines a firm. We consider two cases, distinguished by whether a firm has been fined by the regulator, and show that a rational firm has no incentive to deviate to selling data.

1. A firm has been fined.

   Things are trivial if firm 1 has been fined. Its belief will stay at 1. The consumer will reveal nothing to firm 1 and firm 1 will always sell the consumer data. But, as we will show, firm 1 will not deviate to selling the data in equilibrium, and therefore will not be fined in equilibrium. Now consider the case where firm 2 has been fined and firm 1 has not been fined, which implies that the consumer's belief about firm 2 stays at 1. Denote the current current belief about firm 1 by μ_1. Denote the consumer's belief in the next period after observing signal s and s_1^r by μ_1^{s,s_1^r}.[2] By Bayes' rule,

$$
\begin{aligned}
\mu_1^{y,-} =& \frac{P(s=y, s_1^r=-|firm\ 1\ B)P(firm\ 1\ B)}{P(s=y, s_1^r=-|firm\ 1\ B)P(firm\ 1\ B)+P(s=y, s_1^r=-|firm\ 1\ R)P(firm\ 1\ R)} \\
=& \frac{q(2-q)(1-q_r)\mu_1}{q(2-q)(1-q_r)\mu_1+q(1-\mu_1)},
\end{aligned}
$$

   where $P(s=y, s_1^r=-|firm\ 1\ B)$

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[2] The signal s_2^r does not affect the consumer's belief because the consumer knows that firm 2 is bad type, $\mu_2 = 1$.

$$=q^2(1-q_r)+q(1-q)(1-q_r)+(1-q)q(1-q_r)=q(2-q)(1-q_r),$$

$$P(s=y,s_1^r=-|firm\ 1\ R)=q.$$

One can check that $q_r > (1-q)/(2-q) \Leftrightarrow \mu_1^{y,-} < \mu_1$. One can also see that $\mu_1^{n,-} < \mu_1^{y,-}$. By induction, the consumer's belief after observing signal $s = y$ and $s_1^r = -$ for k consecutive periods is

$$\mu_1^{y,-^k} = \frac{q^k(2-q)^k(1-q_r)^k\mu_1}{q^k(2-q)^k(1-q_r)^k\mu_1 + q^k(1-\mu_1)}, \tag{6}$$

which decreases in k.

We now show that firm 1 will not deviate from the equilibrium of privacy protection under this circumstances. The game is continuous at infinity because of discounting. So, we can use the single-deviation property.

We first consider the case where $\mu_1 \leq \widehat{\mu}$. By not deviating, firm 1 will never be fined by the regulator. Because $\mu_1^{n,-} < \mu_1^{y,-} < \mu_1 \leq \widehat{\mu}$, the belief about firm 1 will stay below $\widehat{\mu}$ regardless of the consumer's signal s. Firm 1's value function is $V_1(\mu_1, 1) = (1-\delta)[(v/2)/(1-\delta)] = v/2$.

By deviating only once in the current period, firm 1 will be fined by the regulator with probability q_r. Its belief will stay at 1 and the consumer will never reveal anything to it. With the complementary probability, the regulator does not fine firm 1. As shown above, the belief about firm 1 will stay below $\widehat{\mu}$ regardless of the consumer's signal s. In this case, firm 1's value function is $V_{1,dev}(\mu_1, 1) = q_r(1-\delta)\left\{D(1-v/t) + \frac{v}{2} + \delta[v^2/2t + D(0)]/(1-\delta)\right\} + (1-q_r)(1-\delta)[D(1-v/t) + (v/2)/(1-\delta)]$. Therefore,

$$V_1(\mu_1,1) - V_{1,dev}(\mu_1,1)$$
$$= q_r v/2 - \delta q_r[v^2/2t + D(0)] - (1-\delta)\{q_r[D(1-v/t) + (v/2)] + (1-q_r)D(1-v/t)\}$$

$$> q_r \Delta u - (1-\delta)\{q_r[D(1-v/t) + (v/2)] + (1-q_r)D(1-v/t)\}$$
$$> 0 \ , \forall \delta > 1 - \frac{q_r \Delta u}{q_r[D(1-v/t) + (v/2)] + (1-q_r)D(1-v/t)}.$$

So, a patient enough rational firm will not deviate in this case.

We then consider the case where $\mu_1 > \widehat{\mu}$. The belief about firm 1 after k period is no greater than $\mu_1^{y,-k}$ if firm 1 does not deviate. In addition,

$$\mu_1^{y,-k} < \widehat{\mu} \Leftrightarrow k > \frac{ln[\widehat{\mu}(1-\mu_1)/\mu_1(1-\widehat{\mu})]}{ln[(2-q)(1-q_r)]}. \tag{7}$$

Let $\widehat{k} := \lceil \frac{ln[\widehat{\mu}(1-\mu_1)/\mu_1(1-\widehat{\mu})]}{ln[(2-q)(1-q_r)]} \rceil$. One can see that the belief about firm 1 after $\widehat{k}$ period will be lower than $\widehat{\mu}$ if firm 1 does not deviate. Therefore, similar to the argument in the $\mu_1 \leq \widehat{\mu}$ case, we have $V_1(\mu_1, 1) \geq (1-\delta)[(v^2/2t)(1+\delta+\delta^2+...+\delta^{\widehat{k}-1}) + \delta^{\widehat{k}}(v/2)/(1-\delta)]$ and $V_{1,dev}(\mu_1,1) \leq q_r(1-\delta)\left\{D(0) + \frac{v^2}{2t} + \delta[v^2/2t + D(0)]/(1-\delta)\right\} + (1-q_r)(1-\delta)[D(0) + (v^2/2t) + \delta(v/2)/(1-\delta)]$.

$$V_1(\mu_1,1) - V_{1,dev}(\mu_1,1)$$
$$> \delta^{\widehat{k}} v/2 - q_r\delta[v^2/2t + D(0)] - (1-q_r)\delta \cdot v/2 - (1-\delta)[D(0) + (v^2/2t)]$$
$$> q_r \Delta u - (1-\delta^{\widehat{k}})[v/2 + D(0) + v^2/2t]$$
$$> 0 \ , \forall \delta > [1 - \frac{q_r \Delta u}{v/2 + D(0) + v^2/2t}]^{1/k}.$$

So, a patient enough rational firm will not deviate in this case.

2. No firm has been fined.

The consumer's belief about each firm will be identical in this case. Denote the current belief about each firm by μ. The game is continuous at infinity because of discounting. So, we can use the single-deviation property.

We first consider the case where $\mu < \widehat{\mu}$. By deviating only once in the current period, firm 1's value function is $V_{1,dev}(\mu) \leq q_r(1-\delta)\left\{D(1-v/t)+\frac{v}{2}+\delta[v^2/2t+D(0)]/(1-\delta)\right\}+$ $(1-q_r)(1-\delta)[D(1-v/t)+(v/2)/(1-\delta)]$. If firm 1 does not deviate, its value function is $V_1(\mu)=\mu V_1(\mu|\text{firm 2 is bad})+(1-\mu)V_1(\mu|\text{firm 2 is good})$.

If firm 2 is good, then no firm sells consumer data without deviation. In such cases, the signal is always $s=n, s_i^r=-$. So, the belief stays below $\widehat{\mu}$ and $V_1(\mu|\text{firm 2 is good})=$ $(1-\delta)(v/2)/(1-\delta)=v/2$. If firm 2 is bad, for any integer k_1, firm 2 will be detected and fined by the regulator with probability $1-(1-q_r)^{k_1}$ within the first k_1 period. The belief about firm 1 after k_1 period is the highest if the consumer receives $s=y$ in each period and $s_2^r=-$ in the first k_1-1 period. Denote this upper bound of the belief about firm 1 after k_1 period by $\bar{\mu}_1(k_1,\mu)$, which is strictly lower than 1. Equation (7) implies that the belief about firm 1 will be lower than $\widehat{\mu}$ after another $k_2(k_1,\mu):=$ $\lceil\frac{ln[\widehat{\mu}(1-\bar{\mu}_1(k_1,\mu))/\bar{\mu}_1(k_1,\mu)(1-\widehat{\mu})]}{ln[(2-q)(1-q_r)]}\rceil$ periods if firm 2 has been fined and $\mu_1 \leq \bar{\mu}_1(k_1,\mu)$ after k_1 periods. Note that $k_2(k_1,\mu)$ does not depend on the discount factor δ.

Firm 1 can always gurantee a flow payoff of $(1-\delta)v^2/2t$, and can keep getting a flow payoff of $(1-\delta)v/2$ after $k_1+k_2(k_1,\mu)$ periods if firm 2 is bad and has been fined within the first k_1 periods. Therefore, firm 1's value function by not deviating is:

$$
\begin{aligned}
&V_1(\mu)\\
=&\mu V_1(\mu|\text{firm 2 is bad})+(1-\mu)V_1(\mu|\text{firm 2 is good})\\
\geq&\mu(1-\delta)\left\{[1-(1-q_r)^{k_1}][\frac{v^2}{2t}(1+\delta+...+\delta^{k_1+k_2(k_1,\mu)-1})+\delta^{k_1+k_2(k_1,\mu)}\frac{v/2}{1-\delta}]+(1-q_r)^{k_1}\frac{v^2/2t}{1-\delta}\right\}+\\
&(1-\mu)(1-\delta)\frac{v/2}{1-\delta},\\
&V_1(\mu)-V_{1,dev}(\mu)\\
>&\left\{\mu[1-(1-q_r)^{k_1}]\delta^{k_1+k_2(k_1,\mu)}+q_r-\mu\right\}\frac{v}{2}-\delta q_r[v^2/2t+D(0)]-(1-\delta)[D(1-v/t)+q_r v/2]\\
>&\left\{\mu[1-(1-q_r)^{k_1}]\delta^{k_1+k_2(k_1,\mu)}+q_r-\mu\right\}\frac{v}{2}-q_r[v^2/2t+D(0)]-(1-\delta)[D(1-v/t)+q_r v/2]\\
=&q_r\Delta u-\frac{\mu v}{2}\{1-[1-(1-q_r)^{k_1}]\delta^{k_1+k_2(k_1,\mu)}\}-(1-\delta)[D(1-v/t)+q_r v/2], \qquad (*)
\end{aligned}
$$

which holds for any $k_1 \in \mathbb{N}_+$. One can see that $q_r\Delta u-\frac{\mu v}{2}\{1-[1-(1-q_r)^{k_1}] \to q_r\Delta u$ as $k_1 \to +\infty$. So, there exists a $\bar{k}_1 \in \mathbb{N}_+$ such that $q_r\Delta u-\frac{\mu v}{2}\{1-[1-(1-q_r)^{\bar{k}_1}] >$

$q_r\Delta u/2$. Plug in $k_1 = \bar{k}_1$ to (∗), we have $V_1(\mu) - V_{1,dev}(\mu) > q_r\Delta u - \frac{\mu v}{2}\{1 - [1 - (1 - q_r)^{\bar{k}_1}]\delta^{\bar{k}_1 + k_2(\bar{k}_1,\mu)}\} - (1-\delta)[D(1-v/t) + q_r v/2]$, which approaches $q_r\Delta u - \frac{\mu v}{2}\{1 - [1-(1-q_r)^{\bar{k}_1}]\} > q_r\Delta u/2$ as $\delta \to 1$. Because $q_r\Delta u - \frac{\mu v}{2}\{1 - [1 - (1 - q_r)^{\bar{k}_1}]\delta^{\bar{k}_1+k_2(\bar{k}_1,\mu)}\} - (1 - \delta)[D(1-v/t)+q_r v/2]$ is continuous, there exists a $\bar{\delta} \in (0, 1)$ such that $V_1(\mu) - V_{1,dev}(\mu) > q_r\Delta u - \frac{\mu v}{2}\{1 - [1 - (1 - q_r)^{\bar{k}_1}]\delta^{\bar{k}_1+k_2(\bar{k}_1,\mu)}\} - (1 - \delta)[D(1 - v/t) + q_r v/2] > q_r\Delta u/4 > 0$ for any $\delta \geq \bar{\delta}$.

So, a patient enough rational firm will not deviate in this case.

The idea for the proof of the case where $\mu_1 > \widehat{\mu}$ is similar.

One can see that the regulation can improve consumer welfare when it sustains the equilibrium of privacy protection. In terms of social welfare, let the sum of consumer welfare and firm welfare be V_w when there is such a regulation and V_{wo} when there is no regulation. The regulation improves social welfare if $V_w - (1-\delta)\sum_{t=0}^{+\infty} \delta^t c_r > V_{wo} \Leftrightarrow c_r < V_w - V_{wo}$. □

Proof of Proposition 8. Consider firm 1 without loss of generality. Suppose there exists an equilibrium in which rational firms never sells the data. Then consumers have identical beliefs about both firms. Denote the corresponding value function by $V(\cdot)$. The same argument as in Proposition 5 implies that the belief μ could be arbitrarily close to 1 at some point with a strictly positive probability. Consider $\mu > \widehat{\mu}$. The same argument as in Proposition 5 also implies that $V(\mu) = (1-\delta)\frac{v^2}{2t} + \delta\left[q\mu V(\mu^y) + (1-q\mu)V(\mu^n)\right]$ and the value function of deviating once in the current period is $V_{dev}(\mu) = (1-\delta)\left(\frac{v^2}{2t} + D(0)\right) + \delta\left[q[1+(1-q)\mu]V(\mu^y) + (1-q)(1-q\mu)V(\mu^n)\right]$.[3] In addition, the upper and lower bounds for $V(\mu^n)$ and $V(\mu^y)$ in the proof of Proposition 5 are still valid under this regulation. So, inequality (5) still holds: $V_{dev}(\mu) - V(\mu) \geq (1-\delta)[D(0) - \delta q(1-q\mu)\frac{v}{2}] - \delta q(1-q\mu)\{\delta[q\mu\frac{v^2}{2t} + (1-q\mu)\frac{v}{2}] - [\frac{v^2}{2t} + D(0)]\}$. The same argument as in Proposition 5 implies that $V_{dev}(\mu) - V(\mu) > 0$

[3] The regulation may change the formula for the value function of deviating once if $\mu < \widehat{\mu}$ because the current-period payoff will change from $(1-\delta)[v/2 + D(1-v/t)]$ to $(1-\delta)[v/2 + D(\min\{\bar{\eta}, 1-v/t\})]$, but does not change the formula for the value function of deviating once if $\mu > \widehat{\mu}$ because the consumer reveals no information in this case.

for μ large enough under conditions (1), (2), and (3). Therefore, a rational firm will sell the data when the belief μ is large enough. A contradiction. □

Proof of Proposition 9. Suppose there exists an equilibrium in which a rational firm never sells the data. The belief about a firm will become 1 if it sold data and the consumer identifies its identity. Before consumers identify any identity of data leakage, the consumers have identical beliefs about both firms. Denote the consumer's signal by n if they did not detect any data sales, by y if they caught any of the sales but did not identify any identity of data leakage, by $y, 1$ if they caught any of the sales and identify firm 1 selling data, by $y, 2$ if they caught any of the sales and identify firm 2 selling data, and by $y, 12$ if they caught any of the sales and identify both firms selling data. We assume without loss of generality that firm 1 is a rational firm and consider two cases.

1. imperfect identification $\phi < 1$

   Consider the circumstance where consumers have not identified any identity of data leakage and the belief about each firm is $\mu > \widehat{\mu}$. Denote by μ^n the belief about both firms after signal n, by μ^y the belief about both firms after signal y, by $\mu^{y,1}$ the belief about firm 2 after signal $y, 1$, by $\mu^{y,2}$ the belief about firm 1 after signal $y, 2$.[4] Then, the value function of firm 1 is $V_1(\mu, \mu) = (1-\delta)\frac{v^2}{2t} + \delta[(1-\mu q)V_1(\mu^n, \mu^n) + \mu q\phi V_1(\mu^{y,2}, 1) + \mu q(1-\phi)V_1(\mu^y, \mu^y)]$. The value function of deviating once in the current period is $V_{1,dev}(\mu) = (1-\delta)[\frac{v^2}{2t} + D(0)] + \delta[(1-q)(1-\mu q)V_1(\mu^n, \mu^n) + \mu q\phi(1-q\phi)V_1(\mu^{y,2}, 1) + q(1-\phi)[1+\mu-\mu q(1+\phi)]V_1(\mu^y, \mu^y) + q\phi(1-\mu q\phi)V_1(1, \mu^{y,1}) + \mu q^2\phi^2 V_1(1, 1)]$. Hence,

$$\begin{aligned} & V_{1,dev}(\mu, \mu) - V_1(\mu, \mu) \\ =& (1-\delta)D(0) + \delta\{q(1-\phi)[1-\mu q(1+\phi)]V_1(\mu^y, \mu^y) + q\phi(1-\mu q\phi)V_1(1, \mu^{y,1}) + \\ & \mu q^2\phi^2 V_1(1, 1) - q(1-\mu q)V_1(\mu^n, \mu^n) - \mu q^2\phi^2 V_1(\mu^{y,2}, 1)\}. \end{aligned} \tag{8}$$

   Because firm 1 gets a stage payoff of at most $v/2$ in equilibrium, $V_1(\mu^{y,2}, 1) \leq v/2$.

[4] The belief about firm 1/firm 2/both firms is 1 after signal $y, 1/y, 2/y, 12$.

Always selling consumer data gives a lower bound on the value function:

$$V_1(\mu^y, \mu^y), V_1(1, \mu^{y,1}), V_1(1,1) \geq (1-\delta)\sum_{k=0}^{+\infty} \delta^k [\frac{v^2}{2t} + D(0)] = \frac{v^2}{2t} + D(0).$$

By Bayes' rule, the consumer's belief about both firms after observing signal y is:

$$\begin{aligned}\mu^y =& \frac{P(signal\ y|firm\ 1\ B)P(firm\ 1\ B)}{P(signal\ y|firm\ 1\ B)P(firm\ 1\ B) + P(signal\ y|firm\ 1\ R)P(firm\ 1\ R)} \\ =& \frac{q(1-\phi)[1+\mu-\mu q(1+\phi)]\cdot\mu}{q(1-\phi)[1+\mu-\mu q(1+\phi)]\cdot\mu + \mu q(1-\phi)\cdot(1-\mu)} \\ =& \frac{1+\mu-\mu q(1+\phi)}{2-\mu q(1+\phi)} \geq 1/2.\end{aligned}$$

Condition (3) ($v/u_b < 2$) $\Rightarrow \widehat{\mu} < 1/2$. $\mu^y \geq 1/2$, $\forall\mu$ implies that the consumer will reveal no information after one signal y, which gives firm 1 a stage equilibrium payoff of $\frac{v^2}{2t}$. In other cases, firm 1 gets a stage payoff of at most $v/2$. So,

$$\begin{aligned}V_1(\mu^n, \mu^n) &\leq (1-\delta)\left[\frac{v}{2} + \sum_{k=1}^{+\infty}\delta^k[\mu q(1-\phi)\frac{v^2}{2t} + [1-\mu q(1-\phi)]\frac{v}{2}]\right] \\ &= (1-\delta)\frac{v}{2} + \delta\left[\mu q(1-\phi)\frac{v^2}{2t} + [1-\mu q(1-\phi)]\frac{v}{2}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}& V_{1,dev}(\mu,\mu) - V_1(\mu,\mu) \\ \geq&(1-\delta)D(0) + \delta\left\{q(1-\phi)[1-\mu q(1+\phi)] + q\phi(1-\mu q\phi) + \mu q^2\phi^2\right\}[\frac{v^2}{2t} + D(0)] - \\ & \delta q(1-\mu q)\left\{(1-\delta)\frac{v}{2} + \delta\left[\mu q(1-\phi)\frac{v^2}{2t} + [1-\mu q(1-\phi)]\frac{v}{2}\right]\right\} - \delta\mu q^2\phi^2\frac{v}{2} \\ =&(1-\delta)[D(0) - \delta q(1-\mu q)\frac{v}{2}] + \\ & \delta\{\left[q(1-\phi)[1-\mu q(1+\phi)] + q\phi(1-\mu q\phi) + \mu q^2\phi^2\right][\frac{v^2}{2t} + D(0)] - \\ & q(1-\mu q)\left\{(1-\delta)\frac{v}{2} + \delta\left[\mu q(1-\phi)\frac{v^2}{2t} + [1-\mu q(1-\phi)]\frac{v}{2}\right]\right\} - \mu q^2\phi^2\frac{v}{2}\} \qquad (9)\end{aligned}$$

With a strictly positive probability, the signal will be y for k consecutive periods, $\forall k$. Denote the belief after k consecutive signal y by μ^{y^k}. One can see that $\mu^y \in (\mu, 1), \forall \mu \in (0,1)$. So, μ^{y^k} strictly increases in k and is bounded by 1. Thus, $\{\mu^{y^k}\}_{k=1}^{+\infty}$ has a limit. Denote the limit by $\mu^{y^{+\infty}}$. We have $(\mu^{y^{+\infty}})^y = \mu^{y^{+\infty}} \Rightarrow \mu^{y^{+\infty}} = 1$. So, μ^{y^k} could be arbitrarily close to 1 with a strictly positive probability. If condition (2) holds ($q(1-q)v/2 < D(0)$), for large enough μ, we have $D(0) - \delta q(1-q\mu)\frac{v}{2} > 0$. So, the term following $(1-\delta)$ in equation (9) is positive. If condition (1) holds ($\Delta u < \overline{\Delta u}$), then $(1-q)(v/2 - v^2/2t) < D(0)$. For large enough μ and δ and small enough ϕ, we have $[q(1-\phi)[1-\mu q(1+\phi)] + q\phi(1-\mu q\phi) + \mu q^2\phi^2]\left[\frac{v^2}{2t} + D(0)\right] - q(1-\mu q)\left\{(1-\delta)\frac{v}{2} + \delta\left[\mu q(1-\phi)\frac{v^2}{2t} + [1-\mu q(1-\phi)]\frac{v}{2}\right]\right\} - \mu q^2\phi^2\frac{v}{2} > 0$. So, the term following δ in equation (9) is positive. Together, we get that $V_{1,dev}(\mu,\mu) - V_1(\mu,\mu) > 0$ for large enough μ and δ and small enough ϕ. One can see that the right-hand side of equation (8) is decreasing in δ. So, $V_{1,dev}(\mu,\mu) - V_1(\mu,\mu) > 0$ for large enough μ and small enough ϕ. Therefore, firm 1 will sell the data if conditions (1), (2), and (3) in Proposition 5 hold and if the probability of identifying the identity of data leakage, ϕ, is low. A contradiction.

2. perfect identification $\phi = 1$

   The game is continuous at infinity because of discounting. So, we can use the single-deviation property. There are four cases.

   (a) No identity of data leakage has been identified, $(\mu_1, \mu_2) = (\mu, \mu)$ and $\mu \leq \widehat{\mu}$.

   Firm 1's value function of not deviating is $V_1(\mu,\mu) = (1-\delta)v/2 + \delta[(1-\mu q)V_1(\mu^n, \mu^n) + \mu q V_1(\mu^{y,2}, 1)]$. The value function of deviating once in the current period is $V_{1,dev}(\mu) = (1-\delta)[v/2 + D(1-v/t)] + \delta[(1-q)(1-\mu q)V_1(\mu^n,\mu^n) + (1-q)\mu q V_1(\mu^{y,2}, 1) + q(1-\mu q)V_1(1, \mu^{y,1}) + \mu q^2 V_1(1,1)]$. By Bayes' rule, $\mu^n = \mu^{y,2} = (1-q)\mu/(1-q\mu) < \mu \leq \widehat{\mu}$. Therefore, the consumer's belief about firm 1 is always lower than $\widehat{\mu}$ if firm 1 does not deivate. So, $V_1(\mu^n,\mu^n) = V_1(\mu^{y,2}, 1) = v/2$. One can also see that

$V_1(1, \mu^{y,1}) = V_1(1,1) = v^2/2t + D(0)$.

Hence, $V_1(\mu, \mu) - V_{1,dev}(\mu, \mu) = -(1-\delta)D(1-v/t) + \delta q \Delta u$, which is positive for δ large enough. Therefore, firm 1 does not deviate.

(b) Firm 2 has been identified as a bad type, $(\mu_1, \mu_2) = (\mu_1, 1)$ and $\mu_1 > \widehat{\mu}$.

If firm 1 does not deviate, then the only possible signals are n and $y, 2$. By Bayes' rule, $\mu_1^n = \mu_1^{y,2} = (1-q)\mu_1/(1-q\mu_1)$.[5] Denote the belief after k periods by $\mu_{1,k}$. One can see that $\mu_{1,k}$ strictly decreases in k and is bounded by 0. Thus, $\{\mu_{1,k}\}_{k=1}^{+\infty}$ has a limit. Denote the limit by $\mu_{1,+\infty}$. We have $(\mu_{1,+\infty})^n = \mu_{1,+\infty} \Rightarrow \mu_{1,+\infty} = 0$. Therefore, there exists $\widehat{k} \in \mathbb{N}_+$ such that the belief about firm 1 will be lower than $\widehat{\mu}$ after k periods, for any $k \geq \widehat{k}$.

Firm 1's value function of not deviating is

$$V_1(\mu_1, 1) \geq (1-\delta)\sum_{j=0}^{\widehat{k}} \delta^j \frac{v^2}{2t} + (1-\delta)\delta^{\widehat{k}+1}\sum_{j=0}^{+\infty} \delta^j \frac{v}{2} = (1-\delta^{\widehat{k}+1})\frac{v^2}{2t} + \delta^{\widehat{k}+1}\frac{v}{2} \quad (10)$$

The value function of deviating once in the current period is

$$V_{1,dev}(\mu_1, 1) \leq (1-\delta)[\frac{v^2}{2t} + D(0)] + \delta\{q[\frac{v^2}{2t} + D(0)] + (1-q)\frac{v}{2}\}$$

Hence,

$$\begin{aligned} & V_1(\mu_1, 1) - V_{1,dev}(\mu_1, 1) \\ \geq & -(1-\delta)D(0) + (1-\delta^{\widehat{k}+1})\frac{v^2}{2t} + \delta^{\widehat{k}+1}\frac{v}{2} - \delta q[\frac{v^2}{2t} + D(0)] - \delta(1-q)\frac{v}{2} \\ = & -(1-\delta)D(0) + (1-\delta^{\widehat{k}+1})\frac{v^2}{2t} - \delta(1-\delta^{\widehat{k}})\frac{v}{2} + \delta q \Delta u \\ \geq & -(1-\delta)D(0) - \delta(1-\delta^{\widehat{k}})\frac{v}{2} + \delta q \Delta u, \end{aligned}$$

[5] We abuse notation a bit here for simplicity: we denote by $\mu_1^{y,2}$ the belief about firm 1 after signal $y, 2$ when the previous belief is $(\mu_1, 1)$, whereas we have defined by $\mu^{y,2}$ the belief about firm 1 after signal $y, 2$ when the previous belief is (μ, μ). The notation may be justified by observing that $\mu_1^{y,2} = \mu^{y,2}$ if $\mu_1 = \mu$.

which is positive for δ large enough. Therefore, firm 1 does not deviate.

(c) Firm 2 has been identified as a bad type, $(\mu_1, \mu_2) = (\mu_1, 1)$ and $\mu_1 \leq \widehat{\mu}$.

Firm 1's value function of not deviating is $V_1(\mu_1, 1) = v/2$. The value function of deviating once in the current period is $V_{1,dev}(\mu_1, 1) = (1-\delta)[v/2 + D(1 - v/t)] + \delta[q(v^2/2t + D(0)) + (1-q)v/2]$. Hence, $V_1(\mu_1, 1) - V_{1,dev}(\mu_1, 1) = -(1-\delta)D(1 - v/t) + \delta q \Delta u$, which is positive for δ large enough. Therefore, firm 1 does not deviate.

(d) No identity of data leakage has been identified, $(\mu_1, \mu_2) = (\mu, \mu)$ and $\mu > \widehat{\mu}$.

Firm 1's value function of not deviating is $V_1(\mu, \mu) = (1-\delta)v^2/2t + \delta[(1 - \mu q)V_1(\mu^n, \mu^n) + \mu q V_1(\mu^{y,2}, 1)]$. The value function of deviating once in the current period is $V_{1,dev}(\mu, \mu) = (1-\delta)[v^2/2t + D(0)] + \delta[(1-q)(1-\mu q)V_1(\mu^n, \mu^n) + (1-q)\mu q V_1(\mu^{y,2}, 1) + q(1-\mu q)V_1(1, \mu^{y,1}) + \mu q^2 V_1(1, 1)] = (1-\delta)[v^2/2t + D(0)] + \delta\{(1-q)(1-\mu q)V_1(\mu^n, \mu^n) + (1-q)\mu q V_1(\mu^{y,2}, 1) + q[v^2/2t + D(0)]\}$.

So, $V_1(\mu, \mu) - V_{1,dev}(\mu, \mu) = -(1-\delta)D(0) + \delta q(1-\mu q)V_1(\mu^n, \mu^n) + \delta \mu q^2 V_1(\mu^{y,2}, 1) - \delta q[v^2/2t + D(0)]$. Because $\mu^{y,2} = (1-q)\mu/(1-q\mu) < \mu$, the same argument as in equation (10) implies that $V_1(\mu, 1) \geq (1 - \delta^{\widehat{k}+1})v^2/2t + \delta^{\widehat{k}+1}v/2$. Also, $V_1(\mu^n, \mu^n) \geq v^2/2t + D(0)$. Therefore,

$$\begin{aligned} V_1(\mu, \mu) - V_{1,dev}(\mu, \mu) \geq& -(1-\delta)D(0) + \delta\mu q^2[(1 - \delta^{\widehat{k}+1})\frac{v^2}{2t} + \delta^{\widehat{k}+1}\frac{v}{2} - \frac{v^2}{2t} - D(0)] \\ =& -(1-\delta)D(0) + \delta\mu q^2[\delta^{\widehat{k}+1}\Delta u - (1 - \delta^{\widehat{k}+1})D(0)], \end{aligned}$$

which is positive for δ large enough. Therefore, firm 1 does not deviate.

One can see that both consumer welfare and social welfare improves from the main model if $\phi = 1$.

□

Proof of Proposition 10. Suppose there exists an equilibrium in which a rational firm never sells the data without being falsely detected. Denote the current belief about the firm by μ. Consider the incentive of a rational firm.

1. Consumers are not aware of the possibility of false-positive signals, and update their belief in the same way as in the main model.

   According to the proof of Corolloary 2, $\mu^n = P(typeB|s = n) = (1-q)\mu/(1-q\mu)$, $\mu^y = P(typeB|s = y) = 1$. The game is continuous at infinity because of discounting. So, we can use the single-deviation property. There are two cases.

   (a) $\mu \leq \widehat{\mu}$. The value function of the equilibrium strategy is: $V(\mu_t) = (1-\delta)\frac{v}{2} + \delta[(1-q')V(\mu^n) + q'V(\mu^y)]$. The value function of deviating once in the current period is (assuming the firm sells data when the belief is 1, which maximizes the payoff): $V_{dev}(\mu) = (1-\delta)[\frac{v}{2} + D(1-v/t)] + \delta[(1-q)V(\mu^n) + qV(\mu^y)]$.

$$V_{dev}(\mu) - V(\mu) = (1-\delta)D(1-v/t) - (q-q')\delta[V(\mu^n) - V(\mu^y)] \quad (11)$$

$$V(\mu^y) = v^2/2t + D(0)$$

$$V(\mu^n) = (1-\delta)v/2 + \delta[q'(v^2/2t + D(0)) + (1-q')V(\mu^{nn})]$$

$$= (1-\delta)v/2 + \delta[q'(v^2/2t + D(0)) + (1-q')V(\mu^n)] \quad (12)$$

$$\Rightarrow V(\mu^n) = \frac{(1-\delta)v/2 + \delta q'[v^2/2t + D(0)]}{1-\delta(1-q')} \quad (13)$$

$$\Rightarrow V(\mu^n) - V(\mu^y) = \frac{1-\delta}{1-\delta(1-q')}\Delta u$$

$$\overset{(11)}{\Rightarrow} V_{dev}(\mu) - V(\mu) = (1-\delta)\left[D(1-v/t) - \frac{\delta(q-q')\Delta u}{1-\delta(1-q')}\right]$$

$$\Rightarrow V_{dev}(\mu) - V(\mu) < 0 \Leftrightarrow \delta > \delta_1 := \frac{D(1-v/t)}{(q-q')\Delta u + (1-q')D(1-v/t)},$$

   which is less than 1 if $q' < q'_1 := \dfrac{q\Delta u}{D(1-v/t) + \Delta u}$.

   Equation (12) holds because both μ^n and μ^{nn} are lower than $\widehat{\mu}$, and a single signal $s = y$ moves the belief to 1 permanently, and thereby $V(\mu^{nn}) = V(\mu^n)$.

   Therefore, a rational firm will not deviate if the discount factor is high enough,

$\delta > \delta_1$, and the false detection rate is low enough, $q' < q'_1$.

(b) $\mu > \widehat{\mu}$. Similar to the previous case, we have

$$V_{dev}(\mu) - V(\mu) = (1-\delta)D(0) - (q-q')\delta[V(\mu^n) - V(\mu^y)]. \tag{14}$$

According to Corolloary 2, $\mu^{n^k} = (1-q)^k\mu/[(1-q)^k\mu + 1 - \mu]$, which implies that $\mu^{n^k} < \widehat{\mu} \Leftrightarrow k > ln[\widehat{\mu}(1-\mu)/\mu(1-\widehat{\mu})]/ln(1-q)$. Let $\widehat{k}(\mu) = \lceil ln[\widehat{\mu}(1-\mu)/\mu(1-\widehat{\mu})]/ln(1-q)\rceil$. One can see that the belief after $\widehat{k}(\mu)$ period will be lower than $\widehat{\mu}$ if the rational firm does not deviate and there is no false detection. Because the belief either keeps decreasing or jumps to 1 and stays there forever, we have $\widehat{k}(\mu) \leq \widehat{k}(\mu_0)$. Let $\widehat{k}_0 = \widehat{k}(\mu_0)$.

$$\begin{aligned}
V(\mu^n) \geq& (1-\delta)\frac{v^2}{2t}(1+\delta+...+\delta^{\widehat{k}_0}) + [1-(1-q')^{\widehat{k}_0+1}]\delta^{\widehat{k}_0+1}[\frac{v^2}{2t} + D(0)] + \\
& (1-q')^{\widehat{k}_0+1}\delta^{\widehat{k}_0+1}\left\{(1-\delta)\frac{v}{2} + \delta[(1-q')V(\mu^{n^{\widehat{k}_0+2}}) + q'\left(\frac{v^2}{2t} + D(0)\right)]\right\} \\
V(\mu^y) =& v^2/2t + D(0) \\
\Rightarrow \quad & [V(\mu^n) - V(\mu^y)]/(1-\delta) \\
\geq& (1-q')^{\widehat{k}_0+1}\delta^{\widehat{k}_0+1}\Delta u + \frac{(1-q')^{\widehat{k}_0+2}\delta^{\widehat{k}_0+2}\Delta u}{1-\delta(1-q')} - (1+\delta+...+\delta^{\widehat{k}_0})D(0) \\
\geq& (1-q')^{\widehat{k}_0+1}\delta^{\widehat{k}_0+1}\Delta u + \frac{(1-q')^{\widehat{k}_0+2}\delta^{\widehat{k}_0+2}\Delta u}{1-\delta(1-q')} - (\widehat{k}_0+1)D(0) \\
\overset{(14)}{\Rightarrow} \quad & [V_{dev}(\mu) - V(\mu)]/(1-\delta) \\
\leq& D(0) - (q-q')\delta\left[(1-q')^{\widehat{k}_0+1}\delta^{\widehat{k}_0+1}\Delta u + \frac{(1-q')^{\widehat{k}_0+2}\delta^{\widehat{k}_0+2}\Delta u}{1-\delta(1-q')} - (\widehat{k}_0+1)D(0)\right].
\end{aligned}$$

Denote the above upper bound of $[V_{dev}(\mu) - V(\mu)]/(1-\delta)$ by $J(q', \delta)$. One can see that $J(q', \delta)$ is continous, increases in q', decreases in δ, and approaches $I(q') := D(0) - (q-q')[(1-q')^{\widehat{k}_0+1}\Delta u + [(1-q')^{\widehat{k}_0+2}\Delta u]/q' - (\widehat{k}_0+1)D(0)]$ as $\delta \to 1$. Because $I(q') \to -\infty$ as $q' \to 0^+$, there exists a $q'_2 > 0$ such that $I(q') < -2$, $\forall q' \leq q'_2$.

Consequently, there exists $\delta_2 < 1$ such that $J(q'_2, \delta) < -1$, $\forall \delta \geq \delta_2$. Because $J(q', \delta)$ increases in q', $J(q', \delta) \leq J(q'_2, \delta)$, $\forall q' \leq q'_2$. Therefore, $J(q', \delta) < -1 < 0$ if $\delta \geq \delta_2$ and $q' \leq q'_2$.[6]

Therefore, a rational firm will not deviate if the discount factor is high enough and the false detection rate is low enough.

In sum, let $\widehat{\delta} = \max\{\delta_1, \delta_2\}$, $\widehat{q}' = \max\{q'_1, q'_2\}$. Then, there exists an equilibrium in which a rational firm never sells the data without being falsely detected if $\delta \geq \widehat{\delta}$ and $q' \leq \widehat{q}'$.

2. Consumers are aware of the possibility of false-positive signals and the likelihood of false detection, and update their belief accordingly.

By Bayes' rule,

$$\begin{aligned}
\mu^n = P(typeB|s=n) =& \frac{P(s=n|typeB)P(typeB)}{P(s=n|typeB)P(typeB) + P(s=n|typeR)P(typeR)} \\
=& \frac{(1-q)\mu}{(1-q)\mu + (1-q')(1-\mu)}, \\
\mu^y = P(typeB|s=y) =& \frac{P(s=y|typeB)P(typeB)}{P(s=y|typeB)P(typeB) + P(s=y|typeR)P(typeR)} \\
=& \frac{q\mu}{q\mu + q'(1-\mu)}.
\end{aligned}$$

By induction, the belief after observing k consecutive signal y is $\mu^{y^k} = q^k\mu/[q^k\mu + q'^k(1-\mu)]$, which increases in k. Hence,

$$\mu^{y^k} < \widehat{\mu} \Leftrightarrow k < \frac{ln[\mu(1-\widehat{\mu})/\widehat{\mu}(1-\mu)]}{ln(q'/q)}.$$

Let $\widehat{k}(\mu) := \lfloor \frac{ln[\mu(1-\widehat{\mu})/\widehat{\mu}(1-\mu)]}{ln(q'/q)} \rfloor$. One can see that the belief within the first $\widehat{k}(\mu)$ periods is always lower than $\widehat{\mu}$ along any possibile history.

Consider any belief $\mu < \widehat{\mu}$. The value function of a rational firm is $V(\mu) = (1-\delta)v/2+$

[6] Note that the values of q'_2 and δ_2 do not depend on each other.

$\delta[q'V(\mu^y) + (1-q')V(\mu^n)]$. The value function of deviating once in the current period is $V_{dev}(\mu) = (1-\delta)[v/2 + D(1-v/t)] + \delta[qV(\mu^y) + (1-q)V(\mu^n)]$. Therefore,

$$V_{dev}(\mu) - V(\mu) = (1-\delta)D(1-v/t) - (q-q')\delta[V(\mu^n) - V(\mu^y)] \tag{15}$$

$$V(\mu^y) \geq (1-\delta)\frac{v}{2}[1+\delta+...+\delta^{\widehat{k}(\mu)}] = (1-\delta^{\widehat{k}(\mu)+1})\frac{v}{2}$$

$$V(\mu^n) \leq v/2$$

$$\Rightarrow V(\mu^n) - V(\mu^y) \leq \delta^{\widehat{k}(\mu)+1}\frac{v}{2}$$

$$\overset{(15)}{\Rightarrow} V_{dev}(\mu) - V(\mu) \geq (1-\delta)D(1-v/t) - (q-q')\delta^{\widehat{k}(\mu)+2}\frac{v}{2},$$

$$\Rightarrow V_{dev}(\mu) - V(\mu) > 0 \Leftrightarrow \widehat{k}(\mu) > ln\frac{2(1-\delta)D(1-v/t)}{(q-q')v}/ln\delta - 2 \tag{16}$$

Because $\widehat{k}(\mu)$ increases in μ and $\lim_{\mu\to 0^+}\widehat{k}(\mu) = +\infty$, for any $\delta \in (0,1)$, there exists a $\mu \in (0, \widehat{\mu})$ such that condition (16) holds. Hence, the firm has an incentive to deviate and there does not exist an equilibrium in which a rational firm never sells the data.

□